2020
MATHEMATICS
[HONOURS]
Paper : II

Full Marks : 100 Time : 4 Hours

The figures in the right-hand margin indicate marks.
Symbols, notations have their usual meanings.

GROUP-A
(Differential Calculus)
[Marks : 35]

1. Answer any **three** questions: $1\times3=3$

a) If $u = xyf\left(\frac{y}{x}\right)$, then find $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$.

b) Find $\lim\limits_{x\to0+}\frac{1}{e^{\frac{1}{x}}+1}$.

c) Evaluate the following limit (if exist):

$$\lim_{x\to0}\frac{3x+|x|}{7x-5|x|}.$$

d) Find the nature of discontinuity of the function

$$f(x)=\begin{cases}(x+1)\sin\frac{1}{x} & , \quad x\in(-1, 1)\\ 0 & , \quad \text{otherwise}\end{cases}$$

at x=1.

e) Give an example of a function which is continuously at a point but not differentiable at that point.

2. Answer any **two** questions: $2\times2=4$

a) If $y = x^{n-1}\log x$ then prove that $y_n = \frac{(n-1)!}{x}$.

b) Prove that the curve $r^n = a^n\cos n\theta$ and $r^n = b^n\sin n\theta$ cut orthogonally.

c) Find the derived function f' corresponding to the function $f:[0, 3]\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}x & , \quad \text{when} \quad 0\le x\le1\\ 2-x^2 & , \quad \text{when} \quad 1<x<2\\ x-x^2 & , \quad \text{when} \quad 2\le x\le3.\end{cases}$$

3. Answer any **three** questions: 6×3=18

a) Let $f:[a, b] \to \mathbb{R}$ be such that $f''(x)$ exists in $[a, b]$ and $f'(a) = f'(b)$. Prove that

$$f\left(\frac{a+b}{2}\right) = \frac{1}{2}\left[f(a)+f(b)\right] + \frac{1}{8}(b-a)^2 f''(c)$$

for some $c \in (a, b)$. 6

b) i) Find all asymptotes of the following curve:

$$y = \frac{x^2 - x - 2}{x - 2}$$

ii) Find a and b such that

$$\lim_{x \to 0} \frac{ae^x + be^{-x} + 2\sin x}{\sin x + x\cos x} = 2. \qquad 3+3$$

c) If $y = \log\left[x + \sqrt{1+x^2}\right]$, then prove that

$y_{2n}(0) = 0$ and $y_{2n+1}(0) = (-1)^n . 1^2 . 3^2 . 5^2 ... (2n-1)^2$.

6

d) Define pedal equation of a curve. Show that the pedal equation of the curve $r\cos\left(\frac{\sqrt{a^2-b^2}}{a}\theta\right) = \sqrt{a^2-b^2}$ is $p\sqrt{b^2+r^2} = ar$.

1+5

e) i) Find $\frac{\partial z}{\partial x}$ for the following function:

$$x^2 \sin(2y - 5z) = 1 + y\cos(6zx)$$

ii) A function f is defined on $[0, 1]$ by $f(0) = 1$ and

$f(x) = 0$ if x be irrational

$= \frac{1}{n}$, if $x = \frac{m}{n}$ where m, n are positive integers prime to each other.

Prove that f is continuous at every irrational point in $[0, 1]$ and discontinuous at every rational point in $[0, 1]$. 2+4

4. Answer any **one** question: 10×1=10

a) i) If ρ_1, ρ_2 be the radii of curvature at the end of a focal chord of the parabola $y^2 = 4ax$ then show that

$$\rho_1^{-\frac{2}{3}} + \rho_2^{-\frac{2}{3}} = (2a)^{-\frac{2}{3}}.$$

ii) Find the multiple points of the curve

$$x^4 - 4ax^3 - 4ay^3 + 4a^2x^2 + 3a^2y^2 - a^4 = 0.$$

iii) Discuss the nature of discontinuity at x=1 of the function

$$f(x) = \operatorname*{Lt}_{n \to \infty} \frac{\log(2+x) - x^{2n}\sin x}{1 + x^{2n}}.$$

iv) If f is monotonically increasing function on [a, b] and a<c<b, then show that $f(c+o)$ exists. 3+2+2+3=10

b) i) If $x^2+y^2+z^2-2xyz=1$ then show that

$$\frac{dx}{\sqrt{1-x^2}}+\frac{dy}{\sqrt{1-y^2}}+\frac{dz}{\sqrt{1-z^2}}=0.$$

ii) If $u=\log\left(x^3+y^3+z^3-3xyz\right)$ then show that $u_{xx}+u_{yy}+u_{zz}=-\frac{3}{(x+y+z)^2}$.

iii) If $f(x)=\begin{cases}1 & \text{when } x \text{ is rational}\\ 0 & \text{when } x \text{ is irrational,}\end{cases}$

state with reasons, which of the following statement is true:

p. f is continuous at rational points, but discontinuous at irrational points.

q. f is continuous at irrational points and discontinuous at rational points.

r. f is continuous every where

s. f is discontinuous everywhere.

3+3+4=10

GROUP-B

(Integral Calculus)

[Marks : 25]

5. Evaluate any **one** of the following: 3×1=3

a) $\int\frac{dx}{(1-x)\sqrt{1-x^2}}$

b) $\int\sin^{-1}\left(\sqrt{\frac{x}{a+x}}\right)dx$

6. Answer any **two** questions: 6×2=12

a) Prove that

$$\int_0^{\frac{\pi}{2}}\frac{dx}{\left(a^2\cos^2 x+b^2\sin^2 x\right)^2}=\frac{\pi}{4}\cdot\frac{a^2+b^2}{a^3.b^3};\ a, b>0.$$

6

b) i) Find

$$\lim_{n\to\infty}\left[\frac{n}{n^2}+\frac{n}{1^2+n^2}+\frac{n}{2^2+n^2}+\ldots+\frac{n}{(n-1)^2+n^2}\right].$$

ii) Evaluate: $\int\left(\sin^{-1}x\right)^4 dx$. 3+3

c) Evaluate: 3+3

i) $\int \frac{dx}{\sin(x-a)\sin(x-b)}$

ii) $\int \frac{x^2+2x+3}{\sqrt{1-x^2}}dx$

7. Answer any **one** question: 10×1=10

a) i) Find the moment of inertia of a thin uniform lamina in the form of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about its axes.

ii) Find the volume of the solid obtained by the revolution of the cissoid $y^2(2a-x)=x^3$ about its asymptote.

iii) If $u_n=\int_0^1 x^n \tan^{-1}x\,dx$, prove that for $n>2$,

$(n+1)u_n+(n-1)u_{n-2}=\frac{\pi}{2}-\frac{1}{n}$. 3+4+3

b) i) If $I_n=\int_0^{\frac{\pi}{2}} x^n \sin x\,dx$, n being a positive integer greater than 1, then show that

$$I_n+n(n-1)I_{n-2}=n\left(\frac{\pi}{2}\right)^{n-1}.$$

Hence find the value of $\int_0^{\frac{\pi}{2}} x^5 \sin x\,dx$.

ii) Find the volume of the solid generated by revolving the cardioide $r=a(1-\cos\theta)$ about the initial line. 6+4

GROUP-C

(Differential Equation–I)

[Marks : 40]

8. Answer any **two** questions: 1×2=2

a) Determine the order and degree of the differential equation

$$\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}=\frac{1}{1+\frac{dy}{dx}}.$$

b) Check whether the differential equation

$$(x^2-2xy-y^2)dx-(x+y)^2dy=0$$

is exact.

c) Determine Wronskian of the functions

e^x, e^{2x}, e^{3x}.

9. Answer any **two** questions: $2\times2=4$

a) Find the orthogonal trajectories of the family of curves $xy = a^2$.

b) Solve the equation: $x\,dy - y\,dx - \cos\frac{1}{x}dx = 0$.

c) Show that the substitution $x = \sin h\,z$ transforms the equation $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 4y$ into $\frac{d^2y}{dz^2} = 4y$.

10. Answer any **four** questions: $6\times4=24$

a) Solve the following differential equation

$$(x+a)^2\frac{d^2y}{dx^2} - 4(x+a)\frac{dy}{dx} + 6y = x.$$

b) Find the eigen values and eigen functions of the equation

$$\frac{d^2y}{dx^2} + \lambda y = 0,\ 0 \le x \le \pi$$

satisfying the boundary conditions y = 0 at $x = 0$ and $\frac{dy}{dx} = 0$ at $x = \pi$.

c) Show that the system of co-axial parabolas $y^2 = 4a(x+a)$ is self orthogonal.

d) Reduce the differential equation $y = 2px - p^2y$ to clairant's form by the substitutions $y^2 = Y$ and $x = X$, and obtain the complete primitive and singular solution, if any.

e) Solve the following differential equation by reducing to it's normal form:

$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2-3)y = e^{x^2}.$$

f) Solve the differential equation

$$\frac{d^3y}{dx^3} + y = e^{2x}\sin x + e^{\frac{x}{2}}\sin\frac{\sqrt{3}}{2}x.$$

11. Answer any **one** question: $10\times1=10$

a) i) Determine whether the equation

$$(1+yz)dx + x(z-x)dy - (1+xy)dz = 0$$

is integrable. Also obtain the integral of the equation, if integrable.

ii) If y_1 and y_2 are two linearly independent integrals of the differential equation

$$\frac{d^2y}{dx^2} + p_1\frac{dy}{dx} + p_2y = 0$$

where p_1, p_2 are functions of x, then show that

$$y_1 \frac{dy_2}{dx} - y_2 \frac{dy_1}{dx} = c\,e^{-\int p_1 dx}$$

where $c(\neq 0)$ is a constant. 4+6=10

b) i) Reduce the differential equation $\sin^2 x \frac{dy}{dx} - 2y = 0$ to exact form and hence solve it.

ii) Solve: $\frac{dx}{dt} + 5x + y = e^t$

$$\frac{dy}{dt} - x + 3y = e^{2t}.$$ 5+5=10

—————